HOMEOMORPHISMS OF BAGPIPES

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Abstract. We investigate the mapping class group of an orientable ω-bounded surface. Such a surface splits, by Nyikos's Bagpipe Theorem, into a union of a bag (a compact surface with boundary) and finitely many long pipes. The subgroup consisting of classes of homeomorphisms fixing the boundary of the bag is a normal subgroup and is a homomorphic image of the product of mapping class groups of the bag and the pipes.

1. Introduction

A quarter of a century ago Nyikos gave in his Bagpipe Theorem a decomposition of an ω-bounded surface $F = K \cup (\bigcup_{i=1}^{n} P_i)$, where K is a compact surface with finitely many holes and $P_1, \dots P_n$ is a collection of long pipes attached at these holes. This decomposition makes feasible a partial analysis of the mapping class group of such a surface. The mapping class group $\mathcal{M}(F)$ is the quotient of the group of homeomorphisms of F by the normal subgroup of those isotopic to the identity. In this paper we will confine our consideration to orientable manifolds where convention dictates that the homeomorphisms are orientation-preserving.

We call a connected Hausdorff space each point of which has a neighbourhood homeomorphic to $\mathbb{R}^n$ an *n-manifold* or just *manifold*. A 2-manifold will be called a *surface*. We denote the identity homeomorphism by $\mathbf{1}$. The *open long ray* and *long line* will be denoted respectively by $\mathbb{L}_+$ and $\mathbb{L}$. A topological space X is called *ω-bounded*, [8, p. 662], provided that every countable subset of X has compact closure. The term *long pipe* is defined precisely in Section 3 but essentially it is the union of an increasing ω_1-sequence of open sets homeomorphic to $\mathbb{S}^1 \times [0, \infty)$. Recall that two homeomorphisms $h_0, h_1 : X \to X$ are *isotopic* provided that there is a continuous function $H : X \times [0, 1] \to X$ such that if $H_t : X \to X$ is defined for each $t \in [0, 1]$ by $H_t(x) = H(x, t)$ then H_t is a homeomorphism and $H_0 = h_0$ and $H_1 = h_1$. The map H is called an *isotopy* from h_0 to h_1. If h_0 is isotopic to h_1 then we write $h_0 \cong h_1$.

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Our first main result, Theorem 9, shows that for F as above, any homeomorphism $g : F \to F$ is isotopic to a homeomorphism h which leaves the bag K invariant and permutes the pipes. Consequently some finite power of h, say h^m, leaves both the bag and each pipe invariant. We show that h could have been chosen so that h^m fixes $\underset{\cdot}{K}$ pointwise. Turning to the subgroup $\mathcal{N}(F) < \mathcal{M}(F)$ consisting of those classes of homeomorphisms leaving bag and pipes invariant, we find in Theorem 12 that $\mathcal{N}(F)$ is a normal subgroup and is the homomorphic image of the direct product of the groups $\mathcal{M}(K)$, $\mathcal{M}(P_1)$, ..., $\mathcal{M}(P_n)$.

In Section 2 we consider a fairly general situation involving homeomorphisms and isotopies of Type I spaces, of which ω-bounded surfaces are a special case. A space X is of *Type I*, [8, p. 639], provided that it is the union of an ω_1-sequence $\langle U_\alpha \rangle$ of open subsets such that $\overline{U_\alpha} \subset U_\beta$ and U_α is Lindelöf whenever $\alpha < \beta < \omega_1$. Corollary 5.4 of [8] states that a manifold is ω-bounded if and only if it is countably compact and of Type I.

Section 3 applies the results of Section 2 to ω-bounded surfaces and it is here that we prove Theorem 9. In Section 4 we look specifically at the mapping class groups and there we verify the facts noted above concerning $\mathcal{N}(F)$.

In Section 5 we discuss briefly some possible forms of the mapping class group of a long pipe. This is followed by a short section where we raise some questions for further investigation.

2. Type I Spaces and Homeomorphisms

Theorem 1. *Suppose that X is of Type I, with $X = \cup_{\alpha \in \omega_1} U_\alpha$ as in the definition, and that $h : X \to X$ is a homeomorphism. Then $\{\alpha < \omega_1 \ / \ h(\cup_{\beta<\alpha} U_\beta) = \cup_{\beta<\alpha} U_\beta\}$ is a closed unbounded subset of ω_1.*

Proof. Denote the set by S.

(1) S **is closed.** Suppose that $\langle \alpha_m \rangle$ is an increasing sequence of elements of S with $\alpha_m \uparrow \alpha$. Then

$$\begin{aligned} h\left(\bigcup\nolimits_{\beta<\alpha} U_\beta\right) &= h\left(\bigcup\nolimits_{m=0,\ \beta<\alpha_m}^{\infty} U_\beta\right) \\ &= \bigcup\nolimits_{m=0}^{\infty} h\left(\bigcup\nolimits_{\beta<\alpha_m} U_\beta\right) \\ &= \bigcup\nolimits_{m=0}^{\infty} \bigcup\nolimits_{\beta<\alpha_m} U_\beta \\ &= \bigcup\nolimits_{\beta<\alpha} U_\beta, \end{aligned}$$

so $\alpha \in S$.

(2) S **is unbounded.** Given any $\alpha_0 < \omega_1$, construct an increasing sequence $\langle \alpha_m \rangle$ as follows. Using Lindelöfness of U_{α_m}, given α_m with m even, choose $\alpha_{m+1} > \alpha_m$ so that $h(U_{\alpha_m}) \subset U_{\alpha_{m+1}}$, and given α_m with m odd, choose $\alpha_{m+1} > \alpha_m$ so that $h^{-1}(U_{\alpha_m}) \subset U_{\alpha_{m+1}}$. As an

increasing sequence, $\langle \alpha_m \rangle$ converges, say to α. Moreover

$$\begin{aligned} h\left(\bigcup_{\beta<\alpha} U_\beta\right) &= h\left(\bigcup_{m=0, m \text{ even}}^{\infty} U_{\alpha_m}\right) \\ &= \bigcup_{m=0, m \text{ even}}^{\infty} h(U_{\alpha_m}) \\ &\subset \bigcup_{m=0, m \text{ even}}^{\infty} U_{\alpha_{m+1}} \\ &= \bigcup_{\beta<\alpha} U_\beta \end{aligned}$$

and similarly, by letting m range through the odd integers,

$$h^{-1}\left(\bigcup_{\beta<\alpha} U_\beta\right) \subset \bigcup_{\beta<\alpha} U_\beta,$$

so $\alpha \in S$. Note also that $\alpha > \alpha_0$. ∎

Corollary 2. *Suppose that X is of Type I, with $X = \cup_{\alpha\in\omega_1} U_\alpha$ as in the definition, and that $h_t : X \to X$, $t \in [0,1]$, is an isotopy of homeomorphisms. Then*

$$\left\{\alpha < \omega_1 \ / \ h_t\left(\bigcup_{\beta<\alpha} U_\beta\right) = \bigcup_{\beta<\alpha} U_\beta \text{ for all } t\right\}$$

is a closed unbounded subset of ω_1.

Proof. Choose any countable dense subset $D \subset [0,1]$. By Theorem 1, for each $t \in D$ the set

$$\left\{\alpha < \omega_1 \ / \ h_t\left(\bigcup_{\beta<\alpha} U_\beta\right) = \bigcup_{\beta<\alpha} U_\beta\right\}$$

is a closed unbounded subset of ω_1. As a countable intersection of closed unbounded subsets of ω_1 is closed and unbounded it follows that

$$\left\{\alpha < \omega_1 \ / \ h_t\left(\bigcup_{\beta<\alpha} U_\beta\right) = \bigcup_{\beta<\alpha} U_\beta \text{ for all } t \in D\right\}$$

is a closed unbounded subset of ω_1. As D is dense it follows that

$$\left\{\alpha < \omega_1 \ / \ h_t\left(\bigcup_{\beta<\alpha} U_\beta\right) = \bigcup_{\beta<\alpha} U_\beta \text{ for all } t\right\}$$

is a closed unbounded subset of ω_1. ∎

Corollary 3. *Suppose that M is a metrisable manifold and that $h_t : M \times \mathbb{L}_+ \to M \times \mathbb{L}_+$, $t \in [0,1]$, is an isotopy of homeomorphisms. Then there are $\alpha < \omega_1$ and an isotopy $g_t : M \to M$ such that*

$$h_t(\{x\} \times [\alpha, \omega_1)) = \{g_t(x)\} \times [\alpha, \omega_1)$$

for each $x \in M$.

Proof. Let $D \subset [0,1]$ be a countable dense subset. As a metrisable manifold, M is also separable; say $E \subset M$ is a countable dense subset. For each $t \in [0,1]$ and $x \in M$ consider the function $\theta_{t,x} : \mathbb{L}_+ \to M$ which sends $y \in \mathbb{L}_+$ to the first coordinate of $h_t(x,y)$. By applying [8, Lemma 3.4(iii)], and recalling that a metrisable manifold embeds in euclidean space, see also [2, Lemma 4.3], one can find $\alpha_{t,x} < \omega_1$ so that $\theta_{t,x}|[\alpha_{t,x}, \omega_1)$ is constant. Let $\alpha < \omega_1$ be an upper bound for the countable set $\{\alpha_{t,x} \ / \ t \in D \text{ and } x \in E\}$.

Thus when $t \in D$ and $x \in E$, $\theta_{t,x}(y)$ is independent of $y \in \mathbb{L}_+$ when $y \geq \alpha$. As D and E are dense it follows that $\theta_{t,x}(y)$ is independent of $y \in \mathbb{L}_+$ when $y \geq \alpha$, for all $t \in [0,1]$ and $x \in M$.

Because $M \times \mathbb{L}_+$ is of Type I, by Corollary 2 we may assume that α is such that $h_t(M \times (0,\alpha)) = M \times (0,\alpha)$ for each $t \in [0,1]$. It follows that $h_t(M \times \{\alpha\}) = M \times \{\alpha\}$. Define $g_t : M \to M$ by letting $g_t(x) = \theta_{t,x}(\alpha)$. Then g_t satisfies the requirements. ∎

3. Bagpipes and Homeomorphisms

Definition 4. An *open long pipe* (long pipe in [8, page 662]) is a surface P such that $P = \cup_{\alpha<\omega_1} U_\alpha$, where each U_α is an open subset of P and is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$, such that $\overline{U_\alpha} \subset U_\beta$ and that the boundary of each U_α in U_β is homeomorphic to $\mathbb{S}^1$ whenever $\alpha < \beta$. A *(closed) long pipe* is a surface P with boundary $\mathbb{S}^1$ such that $P = \cup_{\alpha<\omega_1} U_\alpha$, where each U_α is an open subset of P containing the boundary of P and is homeomorphic to $\mathbb{S}^1 \times [0,\infty)$, such that $\overline{U_\alpha} \subset U_\beta$ and that the boundary of each U_α in U_β is homeomorphic to $\mathbb{S}^1$ whenever $\alpha < \beta$.

We prefer to reserve the term *long pipe* for a closed long pipe as in Definition 4 rather than Nyikos's terminology because the pipe then contains a boundary component which may be used to attach the pipe to the boundary of the bag. Note that the equation $P = \cup_{\alpha<\omega_1} U_\alpha$ displays P as a Type I space.

It is easy to show that an ω-bounded long pipe cannot be Lindelöf.

We will call a decomposition of an ω-bounded surface F into $K \cup \left(\bigcup_{i=1}^n P_i \right)$ as in [8, Theorem 5.14], but with closed long pipes, a *Nyikos decomposition.*

Corollary 5. *Suppose the ω-bounded surface F has Nyikos decomposition $K \cup \left(\bigcup_{i=1}^n P_i \right)$ with $P_i = \cup_{\alpha<\omega_1} U_{i,\alpha}$ a decomposition as in the definition of closed long pipe, and that $h_t : F \to F$, $t \in [0,1]$, is an isotopy of homeomorphisms. Then*

$$\left\{\alpha < \omega_1 \ / \ h_t\left(K \cup \left(\bigcup_{i=1,\ \beta<\alpha}^n U_{i,\beta}\right)\right) = K \cup \left(\bigcup_{i=1,\ \beta<\alpha}^n U_{i,\beta}\right) \text{ for all } t\right\}$$

is a closed unbounded subset of ω_1.

Proof. This follows from Corollary 2 because F is of Type I. ∎

Remark 6. The definition of long pipe does not assume that $\cup_{\alpha<\lambda} U_\alpha = U_\lambda$ when $\lambda > 0$ is a limit ordinal (Nyikos calls sequences where this does hold *canonical sequences*, [8, Definition 4.3]). So we cannot replace the equation defining the closed bounded subset of ω_1 in Corollary 5 by the simpler equation $h\left(K \cup \left(\bigcup_{i=1}^n U_{i,\alpha}\right)\right) = K \cup \left(\bigcup_{i=1}^n U_{i,\alpha}\right)$. Example 7 shows

neither the boundary of $\bigcup_{\gamma<\alpha} U_\gamma$ nor the boundary of $P - \overline{\bigcup_{\gamma<\alpha} U_\gamma}$ need be homeomorphic to $\mathbb{S}^1$. It is also worth noting that Nyikos points out in [8, p. 669] that if such 'bad' boundaries show up over and over again in one decomposition then they show up in all decompositions.

Example 7. Consider the set

$$U = \left\{\left(e^{2\pi it}, x\right) \in \mathbb{S}^1 \times \mathbb{R} \ / \ t = 0 \text{ and } x < -1, \right.$$
$$\left. \text{or } 0 < t \leq \tfrac{1}{\pi} \text{ and } x < \sin\tfrac{1}{t}, \text{ or } \tfrac{1}{\pi} \leq t < 1 \text{ and } x < 0\right\},$$

an open subset of $\mathbb{S}^1 \times \mathbb{R}$ which may be expressed as a union $\cup_{n\in\omega} U_n$ where $\overline{U_n} \subset U_{n+1}$ and the boundary of each U_n in U_{n+1} is homeomorphic to $\mathbb{S}^1$. A long pipe P may be constructed so that for each limit ordinal $\lambda > 0$ and each $\alpha < \lambda$ there is a homeomorphism $\varphi : \mathbb{S}^1 \times \mathbb{R} \to U_\lambda$ with $\varphi(\left(\cup_{\alpha<\lambda} U_\alpha\right) = U$. Then the boundaries of $\bigcup_{\gamma<\lambda} U_\gamma$ and $P - \overline{\bigcup_{\gamma<\lambda} U_\gamma}$ are both homeomorphic to the boundary of U and hence not homeomorphic to $\mathbb{S}^1$.

Corollary 8. *Suppose that $h : F \to F$ is a homeomorphism of an ω-bounded surface. Then, apart from compact subsets, h permutes the long pipes of F.*

Thus if we take an appropriate power of h then, apart from compact subsets, long pipes will be preserved set-wise. Hence in order to study the behaviour of homeomorphisms of F we need mainly to study homeomorphisms of long pipes. We can improve on this if we are happy to work within isotopy classes.

Theorem 9. *Suppose the ω-bounded surface F has Nyikos decomposition $K \cup \left(\bigcup_{i=1}^n P_i\right)$ with $P_i = \cup_{\alpha<\omega_1} U_{i,\alpha}$ as in the definition of closed long pipe and that $h : F \to F$ is a homeomorphism. Then there is an isotopy of homeomorphisms $h_t : F \to F$ so that*

1. *$h_0 = h$;*
2. *$h_1(K) = K$.*

Proof. Using Corollary 5 choose any $\alpha < \omega_1$ so that $\alpha > 0$ and

$$h\left(K \cup \left(\bigcup_{i=1,\ \beta<\alpha}^n U_{i,\beta}\right)\right) = K \cup \left(\bigcup_{i=1,\ \beta<\alpha}^n U_{i,\beta}\right).$$

Choose $\gamma, \delta \in \omega_1$ with $\alpha < \gamma < \delta$ so that $h\left(K \cup (\bigcup_{i=1}^n U_{i,\gamma})\right) \subset K \cup (\bigcup_{i=1}^n U_{i,\delta})$. Note that for each $i = 1, \ldots, n$ there is well-defined $\tilde{\imath}$ so that $h\left(\overline{U_{i,\alpha}} - U_{i,\alpha}\right) \subset U_{\tilde{\imath},\delta}$, the correspondence $i \mapsto \tilde{\imath}$ being a bijection. For each $i = 1, \ldots, n$ denote the circle ∂P_i by C_i and let $D_i \subset K$ be a circle disjoint from but close to C_i so that when $i \neq j$ the closed annular region bounded by $C_i \cup D_i$ is disjoint from the closed annular region bounded by $C_j \cup D_j$. Again for each $i = 1, \ldots, n$ let A_i denote the closed region bounded by D_i and $\overline{U_{i,\delta}} - U_{i,\delta}$. For each i the set $A_{\tilde{\imath}}$ is an annulus, by the 2-dimensional Annulus Theorem (see [4, p. 147] or [7, p.91], for example) and $h\left(\overline{U_{i,\alpha}} - U_{i,\alpha}\right)$ is a circle running once around the interior of the annulus. Hence there is a homeomorphism $g_{\tilde{\imath}} : A_{\tilde{\imath}} \to A_{\tilde{\imath}}$ so that $g_{\tilde{\imath}}\left(h\left(\overline{U_{i,\alpha}} - U_{i,\alpha}\right)\right) = C_{\tilde{\imath}}$ and $g_{\tilde{\imath}}$ is $\mathbf{1}$

on the circles $\left(\overline{U_{\tilde{\imath},\delta}} - U_{\tilde{\imath}.\delta}\right) \cup D_{\tilde{\imath}}$. Moreover $\mathbf{1} \cong g_i$ by an isotopy $\bar{g}_t$ which is $\mathbf{1}$ on $\left(\overline{U_{\tilde{\imath},\delta}} - U_{\tilde{\imath}.\delta}\right) \cup D_{\tilde{\imath}}$. We may define h_t to be h except on $\cup_{i=1}^n \mathrm{Int} A_i$ and to be $\bar{g}_t h$ on A_i for each i. ∎

4. Mapping Class Groups

Our first result shows that a homeomorphism leaving the bag invariant is isotopic to a homeomorphism a power of which is the identity on the boundary of the bag.

Proposition 10. *Suppose that $h : K \to K$ is an orientation-preserving homeomorphism of a compact, orientable surface with boundary and m is a positive integer so that h^m is invariant on each boundary component of K. Then there is an isotopy $h_t : K \to K$ so that $h_0 = h$ and h_1^m is the identity on ∂K.*

Proof. It suffices to consider the case where h cycles the boundary components of K. Denote the boundary components by $C_1, \dots, C_m$ and assume that $h(C_i) = C_{i+1}$ for each i (counted modulo m). Let $g_t : C_1 \to C_1$ be an isotopy such that $g_0 = h^m$ and $g_1 = \mathbf{1}$. Firstly define the isotopy h_t on ∂K as follows: $h_t|C_i = h$ when $i < m$ while $h_t|C_m = g_t h^{1-m}$. Note that $h_0 = h$ while h_1^m on C_i, for any $i = 1, \dots, m$, is
$(h_1|C_{i-1}) \dots (h_1|C_1)(h_1|C_m)(h_1|C_{m-1}) \dots (h_1|C_i)$
$= h^{i-1}(g_1 h^{1-m})h^{m-i} = \mathbf{1}$.

Extend h_t over K as follows. By [3, Theorem 2], C_m is collared in K, more precisely, there is an embedding $e : C_m \times [0,1] \to K$ so that $e(x,1) = x$ for each $x \in C_m$. Define the isotopy of embeddings $\varphi_t : K \to K$ by $\varphi_t(e(x,s)) = e(x, (1 - \frac{t}{2})s)$ when $(x,s) \in C_m \times [0,1]$ and φ_t the identity on $K - e(C_m \times (0,1])$. Note that φ_t is well-defined on $e(C_m \times \{0\})$ and that $\varphi_t(K) = K - e(C_m \times (1 - \frac{t}{2}, 1])$. Define

$$h_t(y) = \begin{cases} e(h_{2s+t-2)}(x), s) & \text{if} \quad y = e(x,s) \text{ and } s \geq 1 - \frac{t}{2}, \\ \varphi_t h \varphi_t^{-1}(y) & \text{if} \quad y = e(x,s) \text{ and } s \leq 1 - \frac{t}{2} \\ & \qquad \text{or } y \in K - e(C_m \times (0,1]). \end{cases}$$

If $y = e(x, 1 - \frac{t}{2})$ then
$\varphi_t h \varphi_t^{-1}(y) = \varphi_t h e(x,1) = \varphi_t e(h(x),1) = e(h(x), 1 - \frac{t}{2}) = e(h(x), 0)$
$= e(h_0(x), s) = e(h_{2s+t-2}(x), s)$
so h_t is well-defined. When $s = 1$, $h_t(e(x,1)) = e(h_t(x),1) = h_t(x)$, so h_t really does extend the function h_t already defined. It is routine to show that h_t is an isotopy. It is also routine to show that $h_0 = h$. ∎

Corollary 11. *Let $F = K \cup (\bigcup_{i=1}^n P_i)$ be a Nyikos decomposition of an orientable ω-bounded surface and $h : F \to F$ an orientation preserving homeomorphism such that $h(K) = K$ and $h(P_i) = P_i$ for each i. Then h is isotopic to a homeomorphism which is the identity on ∂K.*

Proof. If we restrict h to K then we may obtain the required isotopy on K from Proposition 10 with $m = 1$. The same procedure as in the proof of Proposition 10 enables us to extend the isotopy over each pipe P_i. ∎

We use the following notation, in which $[h]$ denotes the equivalence class of h under isotopy (preserving the boundary if applicable) and $F = K \cup (\bigcup_{i=1}^{n} P_i)$ is a Nyikos decomposition of an orientable ω-bounded surface. Recall that we are assuming all homeomorphisms are orientation-preserving.

- $\mathcal{M}(F) = \{[h] \ / \ h : F \to F \text{ is a homeomorphism}\}$;
- $\mathcal{M}(K) = \{[h] \ / \ h : K \to K \text{ is a homeomorphism and } h|\partial K = \mathbf{1}\}$;
- $\mathcal{M}(P_i) = \{[h] \ / \ h : P_i \to P_i \text{ is a homeomorphism and } h|\partial P_i = \mathbf{1}\}$;
- $\mathcal{N}(F) = \{[h] \in \mathcal{M}(F) \ / \ h(K) = K \text{ and } h(P_i) = P_i \text{ for each } i\}$.

Theorem 12. *Let $F = K \cup (\bigcup_{i=1}^{n} P_i)$ be a Nyikos decomposition of an orientable ω-bounded surface and let $\mathcal{M}(F)$ and $\mathcal{N}(F)$ be as above. Then*

- *$\mathcal{N}(F)$ is a normal subgroup of $\mathcal{M}(F)$;*
- *there is an epimorphism $\theta : \mathcal{M}(K) \times \mathcal{M}(P_1) \times \cdots \times \mathcal{M}(P_n) \to \mathcal{N}(F)$.*

Proof. Firstly suppose that $g, h : F \to F$ are homeomorphisms so that $[h] \in \mathcal{N}(F)$. We must show that $[g^{-1}hg] \in \mathcal{N}(F)$. By Theorem 9 we may assume that $g(\partial K) = \partial K$ and by Corollary 11 we may assume that h is the identity on ∂K. Then $g^{-1}hg$ is the identity on ∂K and hence $[g^{-1}hg] \in \mathcal{N}(F)$.

Secondly, define θ as follows. Given $([h_0], [h_1], \ldots, [h_n]) \in \mathcal{M}(K) \times \mathcal{M}(P_1) \times \cdots \times \mathcal{M}(P_n)$ let $h : F \to F$ be the homeomorphism which restricts to h_0 on K and to h_i on P_i for each $i = 1, \ldots, n$ and set $\theta([h_0], [h_1], \ldots, [h_n]) = [h]$. Then θ is a homomorphism. That θ is an epimorphism follows from Corollary 11. ∎

Remark 13. The kernel of the homomorphism θ of Theorem 12 consists of those $(n+1)$-tuples $([h_0], [h_1], \ldots, [h_n])$ for which the homeomorphism h constructed in the proof above is isotopic to the identity. Set $C_i = \partial P_i$ for each i. Then $\partial K = \cup_{i=1}^{n} C_i$. Each homeomorphism h_i $(i > 0)$ may be chosen to be the identity except in a collared neighbourhood (in P_i) of C_i where h_i acts as some number of Dehn twists. The homeomorphism h_0 may then be chosen to be the identity except in a collared neighbourhood (in K) of $\cup_{i=1}^{n} C_i$; in the collared neighbourhood of C_i, h_0 acts by reversing the Dehn twists which h_i applied on the opposite side of C_i.

Remark 14. The mapping class group $\mathcal{M}(K)$ is well known. In particular if K has genus γ and has n boundary components then [5, Theorem 1] gives $2\gamma + 2n - 1$ specific generators for this group. Each of these generators is determined by a Dehn twist around a closed curve which may circle a boundary component, traverse a handle or loop around several handles and/or boundary components. Using these generators we may construct interesting homeomorphisms of ω-bounded surfaces.

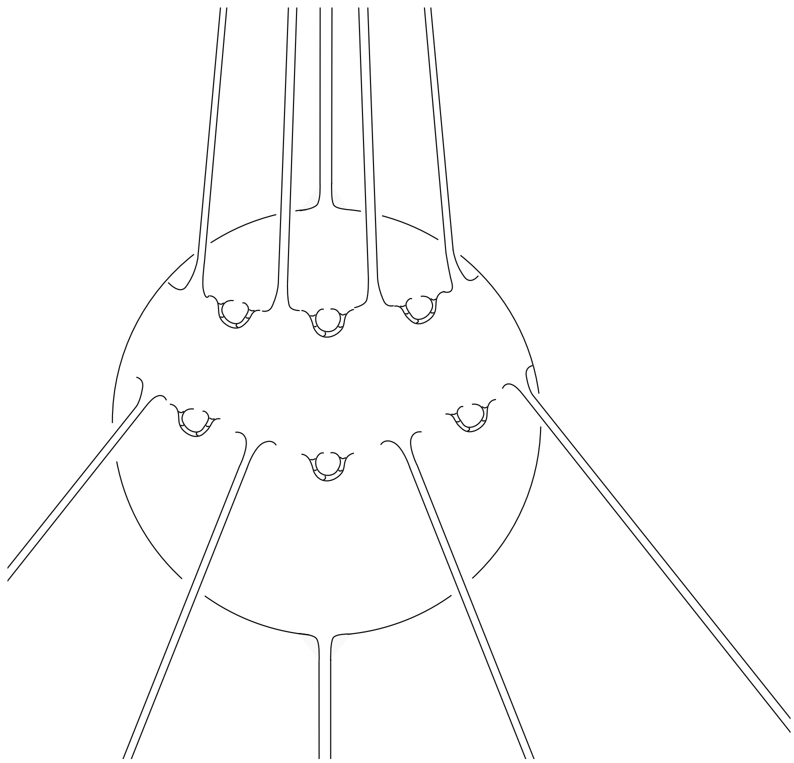

FIGURE 1. A bristly sphere.

Example 15. Replace a small polar cap on the 2-sphere $\mathbb{S}^2$ by a long pipe. Whereas all orientation preserving homeomorphisms of $\mathbb{S}^2$ are isotopic to the identity it may be that not all homeomorphisms of the long pipe are. For example in Proposition 18 below we construct a long pipe supporting a homeomorphism whose mapping class (allowing the boundary to move) has order n for any preassigned positive integer n. As a result we may obtain homeomorphisms, rotations of the remains of the sphere, which are not isotopic to the identity. We could add a further long pipe near the opposite pole

Example 16. We may obtain a surface of higher genus and having more long pipes by spreading n handles and n mutually homeomorphic long pipes around the equator, arranged in such a way that a rotation of the sphere through $\frac{2\pi}{n}$ takes each handle and long pipe to the adjacent handle (respectively long pipe). More bands of n handles or long pipes or both may be distributed along other lines of latitude. Of course the long pipes within a particular band must be mutually homeomorphic. See figure 1.

Example 17. An interesting question is whether there are homeomorphisms a finite power of which are isotopic to the identity: thus the mapping class group has torsion. In the case of the torus $\mathbb{T}^2$ we may look at the group

$GL(2,\mathbb{Z})$ as each isotopy class of homeomorphisms of $\mathbb{T}^2$ may be represented by an element of this group. If an element A of $GL(2,\mathbb{Z})$ is of finite order then its eigenvalues must be roots of the cyclotomic polynomial of degree n which must therefore divide the characteristic polynomial of A and hence have degree 2. It follows that n can be only 1, 2, 3, 4 or 6. In [6, Theorem 2] it is shown that if $A \in GL(2,\mathbb{Z})$ is periodic then A is conjugate to one of the six matrices

- $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which has order 2, switching the coordinates;
- $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, which has order 2, reversing one coordinate;
- $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2$, which has order 2, rotating through $180°$;
- $\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}^2$, which has order 3;
- $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, which has order 4, rotating through $90°$;
- $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$, which has order 6.

Note that some of these matrices have determinant -1 and hence give rise to orientation reversing homeomorphisms. As in the previous example we may add strategically placed long pipes and handles to add to the interest.

5. Long Pipes

It is almost hopeless to attempt to determine the mapping class group of all long pipes in the same way as the mapping class group has been determined for compact surfaces with boundary as in [5]. Indeed, as noted in [8, p. 669], there are $2^{\aleph_1}$ topologically distinct long pipes. In this section we describe a few of these, especially one possessing homeomorphisms of order n for a fixed given $n \in \mathbb{N}$.

The simplest example of a long pipe is $\mathbb{S}^1 \times \mathbb{L}_+$, the product of a circle with the closed long ray. Any orientation preserving homeomorphism is isotopic to the identity.

Before introducing the next example we introduce another concept. Say that a homeomorphism $h : X \to X$ has *isotopy order* n provided that n is the least positive integer for which $h^n = \mathbf{1}$.

The long plane $\mathbb{L}^2$ with an open disc removed is another reasonably simple example of a long pipe. The mapping class group of $\mathbb{L}^2$ has been determined in [1, Theorem 1.3] where it is seen that the isotopy order of any homeomorphism of $\mathbb{L}^2$ (and hence of the long pipe obtained from it) is 1, 2 or 4. This long pipe inspires the next proposition.

Proposition 18. *For every natural number n there is a long pipe P and a homeomorphism $h : P \to P$ whose (isotopy) order is n.*

Proof. Given n construct the long pipe P as follows. Take n copies of the truncated first octant $\{(x, y) \in \mathbb{L}^2 \ / \ x \geq y \geq 0 \text{ and } x \geq 1\}$, say $\mathbb{O}_1, \dots, \mathbb{O}_n$; denote the point of $\mathbb{O}_i$ corresponding to (x, y) in the first octant by $(x, y)_i$. Denote by $\mathbb{O}_i^0$ (resp. $\mathbb{O}_i^1$) the edge of $\mathbb{O}_i$ corresponding to the line $y = 0$ (resp. $y = x$): as noted in [8, Example 3.8] these two subsets behave very differently in $\mathbb{O}_i$ even though each is homeomorphic to the closed long ray. Set $P = \left(\dot{\bigcup}_{i=1}^n \mathbb{O}_i\right) / \sim$, where $\sim$ identifies $(x, x)_i \in \mathbb{O}_i^1$ with $(x, 0)_{i+1} \in \mathbb{O}_{i+1}^0$ for any $i = 1, \dots, n$, where the subscript $i+1$ is modulo n. Define $h : P \to P$ by $h((x, y)_i) = (x, y)_{i+1}$. ∎

Proposition 18 tells us that if there is a homeomorphism of a compact surface with one boundary component which has (isotopy) order n then there is a homeomorphism of a surface with one long pipe having (isotopy) order n. As an example take a surface of genus n where the n handles are spread symmetrically about a central axis like the petals of a simple flower and a small disc centred at one point where the axis cuts the surface is removed.

On the other hand there are long pipes where no homeomorphism has finite isotopy order. The simplest example is $\mathbb{S}^1 \times \mathbb{L}_+$ but $\mathcal{M}(\mathbb{S}^1 \times \mathbb{L}_+)$ has a single element so perhaps is uninteresting. A more interesting example is obtained as in Proposition 18 with $n = 1$; in this case any homeomorphism $h : P \to P$ is fixed (up to isotopy) outside a bounded set so $[h]$ is determined by a number of Dehn twists on an annulus bounded on one side by ∂P and hence $\mathcal{M}(P) \approx \mathbb{Z}$.

6. Questions

Question 19. *Is there a long pipe where no homeomorphism has finite isotopy order?*

Of course in this question we need to be careful what isotopies we allow. The intention here is that isotopies need not be the identity on the boundary.

Question 20. *Suppose that $[h] \in \mathcal{M}(P)$ has finite order, where P is a long pipe. Does it follow that h is isotopic to the identity? In other words is it true that $\mathcal{M}(P)$ has no torsion?*

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